

ABSOLUTE INTEGRAL CLOSURE IN POSITIVE CHARACTERISTIC

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ABSTRACT. Let R be a local Noetherian domain of positive characteristic. A theorem of Hochster and Huneke (1992) states that if R is excellent, then the absolute integral closure of R is a big Cohen-Macaulay algebra. We prove that if R is the homomorphic image of a Gorenstein local ring, then all the local cohomology (below the dimension) of such a ring maps to zero in a finite extension of the ring. There results an extension of the original result of Hochster and Huneke to the case in which R is a homomorphic image of a Gorenstein local ring, and a considerably simpler proof of this result in the cases where the assumptions overlap, e.g., for complete Noetherian local domains.

1. INTRODUCTION

Let R be a commutative Noetherian domain with fraction field K . The *absolute integral closure* of R , denoted R^+ , is the integral closure of R in a fixed algebraic closure \overline{K} of K . This ring was studied by Artin in [3] where among other results he proved that in the case R is Henselian and local, the sum of two primes ideals of R^+ remains prime.

In [6], Hochster and Huneke proved that if (R, \mathfrak{m}) is an excellent local Noetherian domain of positive characteristic $p > 0$, then R^+ is a big Cohen-Macaulay algebra, i.e., every system of parameters in R is a regular sequence on R^+ . The corresponding statement in equicharacteristic 0 is false if the dimension is at least three. Smith [9] further proved that the tight closure of an ideal I generated by parameters is exactly the extension and contraction of I to R^+ : $I^* = IR^+ \cap R$. It is an open question whether the latter equality is true for every ideal I in an excellent Noetherian local domain of positive characteristic. See [1], [2], and [8] for additional work concerning R^+ .

Throughout this paper, R is a commutative Noetherian domain of characteristic $p > 0$ with fraction field K , \overline{K} is a fixed algebraic closure of K , and R^+ is the integral closure of R in \overline{K} . The theorem of Hochster and Huneke in [6] implies the following:

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Let (R, \mathfrak{m}) be an excellent local commutative Noetherian domain of characteristic $p > 0$. Then the natural homomorphism $H_{\mathfrak{m}}^i(R) \rightarrow H_{\mathfrak{m}}^i(R^+)$ is the zero map for every $i < \dim R$.

In fact, as we observe in this paper (see Corollary 2.3), the statement above is basically equivalent to the statement that R^+ is a big Cohen-Macaulay algebra for R .

The main result of this paper, Theorem 2.1 below, states that if R is a homomorphic image of a Gorenstein local ring (though not necessarily excellent), then in fact one can find a *finite* extension ring S , $R \subset S \subset R^+$, such that the map from $H_{\mathfrak{m}}^i(R) \rightarrow H_{\mathfrak{m}}^i(S)$ is zero for all $i < \dim R$. Our proof is independent of the results of [6] and in particular gives a considerably simpler proof of the main result of that paper, with a stronger conclusion, when the assumptions overlap. For example, if R is complete, then it is both excellent and a homomorphic image of a Gorenstein ring. Our proof was in part inspired by the work of Hartshorne and Speiser in [5] and Lyubeznik in [7], concerning the structure of local cohomology modules in positive characteristic.

2. MAIN RESULT

Let R be a commutative ring containing a field of characteristic $p > 0$, let $I \subset R$ be an ideal, and let R' be an R -algebra. The Frobenius ring homomorphism $f : R' \xrightarrow{r \mapsto r^p} R'$ induces a map $f_* : H_I^i(R') \rightarrow H_I^i(R')$ on all local cohomology modules of R' called the action of the Frobenius on $H_I^i(R')$. For an element $\alpha \in H_I^i(R')$ we denote $f_*(\alpha)$ by α^p .

We recall that for a Gorenstein local ring A of dimension n , local duality says that there is an isomorphism of functors $D(\text{Ext}_A^{n-i}(-, A)) \cong H_{\mathfrak{m}}^i(-)$ on the category of finite A -modules, where $D = \text{Hom}_A(-, E)$ is the Matlis duality functor (here E is the injective hull of the residue field of A in the category of A -modules) [4, 11.2.6].

Theorem 2.1. *Let R be a commutative Noetherian local domain containing a field of characteristic $p > 0$, let K be the fraction field of R and let \overline{K} be the algebraic closure of K . Assume R is a surjective image of a Gorenstein local ring A . Let \mathfrak{m} be the maximal ideal of R . Let R' be an R -subalgebra of \overline{K} (i.e. $R \subset R' \subset \overline{K}$) that is a finite R -module. Let $i < \dim R$ be a non-negative integer. There is an R' -subalgebra R'' of \overline{K} (i.e. $R' \subset R'' \subset \overline{K}$) that is finite as an R -module and such that the natural map $H_{\mathfrak{m}}^i(R') \rightarrow H_{\mathfrak{m}}^i(R'')$ is the zero map.*

Proof. Let $n = \dim A$ and let $N = \text{Ext}_A^{n-i}(R', A)$. Since R' is a finite R -module, so is N .

Let $d = \dim R$. We use induction on d . For $d = 0$ there is nothing to prove, so we assume that $d > 0$ and the theorem proven for all smaller dimensions. Let $P \subset R$ be a non-maximal prime ideal. We claim there

exists an R' -subalgebra R^P of \overline{K} (i.e. $R' \subset R^P \subset \overline{K}$) such that R^P is a finite R -module and for every R^P -subalgebra R^* of \overline{K} (i.e. $R^P \subset R^* \subset \overline{K}$) such that R^* is a finite R -module, the image $\mathcal{I} \subset N$ of the natural map $\text{Ext}_A^{n-i}(R^*, A) \rightarrow N$ induced by the natural inclusion $R' \rightarrow R^*$ vanishes after localization at P , i.e. $\mathcal{I}_P = 0$. Indeed, let $d_P = \dim R/P$. Since P is different from the maximal ideal, $d_P > 0$. As R is a surjective image of a Gorenstein local ring, it is catenary, hence the dimension of R_P equals $d - d_P$, and $i < d$ implies $i - d_P < d - d_P = \dim R_P$. By the induction hypothesis applied to the local ring R_P and the R_P -algebra R'_P , which is finite as an R_P -module, there is an R'_P -subalgebra \tilde{R} of \overline{K} , which is finite as an R_P -module, such that the natural map $H_P^{i-d_P}(R'_P) \rightarrow H_P^{i-d_P}(\tilde{R})$ is the zero map. Let $\tilde{R} = R'_P[z_1, z_2, \dots, z_t]$, where $z_1, z_2, \dots, z_t \in \overline{K}$ are integral over R_P . Multiplying, if necessary, each z_j by some element of $R \setminus P$, we can assume that each z_j is integral over R . We set $R^P = R'[z_1, z_2, \dots, z_t]$. Clearly, R^P is an R' -subalgebra of \overline{K} that is finite as R -module.

Now let R^* be both an R^P -subalgebra of \overline{K} (i.e. $R^P \subset R^* \subset \overline{K}$) and a finite R -module. The natural inclusions $R' \rightarrow R^P \rightarrow R^*$ induce natural maps $\text{Ext}_A^{n-i}(R^*, A) \rightarrow \text{Ext}_A^{n-i}(R^P, A) \rightarrow N$. This implies that $\mathcal{I} \subset \mathcal{J}$, where \mathcal{J} is the image of the natural map $\phi : \text{Ext}_A^{n-i}(R^P, A) \rightarrow N$. Hence it is enough to prove that $\mathcal{J}_P = 0$. Localizing this map at P we conclude that \mathcal{J}_P is the image of the natural map $\phi_P : \text{Ext}_{A_P}^{n-i}(\tilde{R}, A_P) \rightarrow \text{Ext}_{A_P}^{n-i}(R'_P, A_P)$ induced by the natural inclusion $R'_P \rightarrow \tilde{R}$ (by a slight abuse of language we identify the prime ideal P of R with its full preimage in A). Let $D_P(-) = \text{Hom}_{A_P}(-, E_P)$ be the Matlis duality functor in the category of R_P -modules, where E_P is the injective hull of the residue field of R_P in the category of R_P -modules. Local duality implies that $D_P(\phi_P)$ is the natural map $H_P^{i-d_P}(R'_P) \rightarrow H_P^{i-d_P}(\tilde{R})$ which is the zero map by construction (note that $i - d_P = \dim A_P - (n - i)$). Since ϕ_P is a map between finite R_P -modules and $D_P(\phi_P) = 0$, it follows that $\phi_P = 0$. This proves the claim.

Since N is a finite R -module, the set of the associated primes of N is finite. Let P_1, \dots, P_s be the associated primes of N different from \mathfrak{m} . For each j let R^{P_j} be an R' -subalgebra of \overline{K} corresponding to P_j , whose existence is guaranteed by the above claim. Let $\overline{R}' = R'[R^{P_1}, \dots, R^{P_s}]$ be the compositum of all the R^{P_j} , $1 \leq j \leq s$. Clearly, \overline{R}' is an R' -subalgebra of \overline{K} (i.e. $R' \subset \overline{R}' \subset \overline{K}$). Since each R^{P_j} is a finite R -module, so is \overline{R}' . Clearly, \overline{R}' contains every R^{P_j} . Hence the above claim implies that $\mathcal{I}_{P_j} = 0$ for every j , where $\mathcal{I} \subset N$ is the image of the natural map $\text{Ext}_A^{n-i}(\overline{R}', A) \rightarrow N$ induced by the natural inclusion $R' \rightarrow \overline{R}'$. It follows that not a single P_j is an associated prime of \mathcal{I} . But \mathcal{I} is a submodule of N , and therefore every associated prime of \mathcal{I} is an associated prime of N . Since P_1, \dots, P_s are all the associated primes of N different from \mathfrak{m} , we conclude that if $\mathcal{I} \neq 0$, then \mathfrak{m} is the only associated prime of \mathcal{I} . Since \mathcal{I} , being a submodule of a finite

R -module N , is finite, and since \mathfrak{m} is the only associated prime of \mathcal{I} , we conclude that \mathcal{I} is an R -module of finite length.

Writing the natural map $\text{Ext}_A^{n-i}(\overline{R}', A) \rightarrow N$ as the composition of two maps $\text{Ext}_A^{n-i}(\overline{R}', A) \rightarrow \mathcal{I} \rightarrow N$, the first of which is surjective and the second injective, and applying the Matlis duality functor D , we get that the natural map $\varphi : H_{\mathfrak{m}}^i(R') \rightarrow H_{\mathfrak{m}}^i(\overline{R}')$ induced by the inclusion $R' \rightarrow \overline{R}'$ is the composition of two maps $H_{\mathfrak{m}}^i(R') \rightarrow D(\mathcal{I}) \rightarrow H_{\mathfrak{m}}^i(\overline{R}')$, the first of which is surjective and the second injective. This shows that the image of φ is isomorphic to $D(\mathcal{I})$ which is an R -module of finite length since so is \mathcal{I} . In particular, the image of φ is a finitely generated R -module. Let $\alpha_1, \dots, \alpha_s \in H_{\mathfrak{m}}^i(\overline{R}')$ generate $\text{Im} \varphi$.

The natural inclusion $R' \rightarrow \overline{R}'$ is compatible with the Frobenius homomorphism, i.e. with the raising to the p th power on R' and \overline{R}' . This implies that φ is compatible with the action of the Frobenius f_* on $H_{\mathfrak{m}}^i(R')$ and $H_{\mathfrak{m}}^i(\overline{R}')$, i.e. $\varphi(f_*(\alpha)) = f_*(\varphi(\alpha))$ for every $\alpha \in H_{\mathfrak{m}}^i(R')$, which, in turn, implies that $\text{Im} \varphi$ is an f_* -stable R -submodule of $H_{\mathfrak{m}}^i(\overline{R}')$, i.e. $f_*(\alpha) \in \text{Im} \varphi$ for every $\alpha \in \text{Im} \varphi$. We finish the proof by applying the following lemma to each element of a finite generating set $\alpha_1, \dots, \alpha_s$ of $\text{Im} \varphi$. Applying Lemma 2.2 below we obtain a \overline{R}' -subalgebra R'_j of \overline{R}' (i.e. $\overline{R}' \subset R'_j \subset \overline{R}'$) such that R'_j is a finite R -module and the natural map $H_{\mathfrak{m}}^i(\overline{R}') \rightarrow H_{\mathfrak{m}}^i(R'_j)$ sends α_j to zero. Let $R'' = R'[R_1, \dots, R_s]$ be the compositum of all the R'_j . Then R'' is an R' -subalgebra of \overline{R}' and is a finite R -module since so is each R'_j . The natural map $H_{\mathfrak{m}}^i(\overline{R}') \rightarrow H_{\mathfrak{m}}^i(R'')$ sends every α_j to zero, hence it sends the entire $\text{Im} \varphi$ to zero. Thus the natural map $H_{\mathfrak{m}}^i(R') \rightarrow H_{\mathfrak{m}}^i(R'')$ is zero. \square

To finish the proof we prove the following lemma, which is closely related to the “equational lemma” in [6] and its modification in [9], (5.3).

Lemma 2.2. *Let R be a commutative Noetherian domain containing a field of characteristic $p > 0$, let K be the fraction field of R and let \overline{K} be the algebraic closure of K . Let I be an ideal of R and let $\alpha \in H_I^i(R)$ be an element such that the elements $\alpha, \alpha^p, \alpha^{p^2}, \dots, \alpha^{p^t}, \dots$ belong to a finitely generated R -submodule of $H_I^i(R)$. There exists an R -subalgebra R' of \overline{K} (i.e. $R \subset R' \subset \overline{K}$) that is finite as an R -module and such that the natural map $H_I^i(R) \rightarrow H_I^i(R')$ induced by the natural inclusion $R \rightarrow R'$ sends α to 0.*

Proof. Let $A_t = \sum_{i=1}^{i=t} R\alpha^{p^i}$ be the R -submodule of $H_I^i(R)$ generated by $\alpha, \alpha^p, \dots, \alpha^{p^t}$. The ascending chain $A_1 \subset A_2 \subset A_3 \subset \dots$ stabilizes because R is Noetherian and all A_t sit inside a single finitely generated R -submodule of $H_I^i(R)$. Hence $A_s = A_{s-1}$ for some s , i.e. $\alpha^{p^s} \in A_{s-1}$. Thus there exists an equation $\alpha^{p^s} = r_1\alpha^{p^{s-1}} + r_2\alpha^{p^{s-2}} + \dots + r_{s-1}\alpha$ with $r_i \in R$ for all i . Let T be a variable and let $g(T) = T^{p^s} - r_1T^{p^{s-1}} - r_2T^{p^{s-2}} - \dots - r_{s-1}T$. Clearly, $g(T)$ is a monic polynomial in T with coefficients in R and $g(\alpha) = 0$.

Let $x_1, \dots, x_d \in R$ generate the ideal I . If M is an R -module, the Čech complex $C^\bullet(M)$ of M with respect to the generators $x_1, \dots, x_d \in R$ is

$$0 \rightarrow C^0(M) \rightarrow \dots \rightarrow C^{i-1}(M) \xrightarrow{d_{i-1}} C^i(M) \xrightarrow{d_i} C^{i+1}(M) \rightarrow \dots \rightarrow C^d(M) \rightarrow 0$$

where $C^0(M) = M$ and $C^i(M) = \bigoplus_{1 \leq j_1 < \dots < j_i \leq d} R_{x_{j_1} \dots x_{j_i}}$, and $H_I^i(M)$ is the i th cohomology module of $C^\bullet(M)$ [4, 5.1.19].

Let $\tilde{\alpha} \in C^i(R)$ be a cycle (i.e. $d_i(\tilde{\alpha}) = 0$) that represents α . The equality $g(\alpha) = 0$ means that $g(\tilde{\alpha}) = d_{i-1}(\beta)$ for some $\beta \in C^{i-1}(R)$. Since $C^{i-1}(R) = \bigoplus_{1 \leq j_1 < \dots < j_{i-1} \leq d} R_{x_{j_1} \dots x_{j_{i-1}}}$, we may write $\beta = (\frac{r_{j_1, \dots, j_{i-1}}}{x_{j_1}^{e_1} \dots x_{j_{i-1}}^{e_{i-1}}})$ where $r_{j_1, \dots, j_{i-1}} \in R$, the integers e_1, \dots, e_{i-1} are non-negative, and $\frac{r_{j_1, \dots, j_{i-1}}}{x_{j_1}^{e_1} \dots x_{j_{i-1}}^{e_{i-1}}} \in R_{x_{j_1} \dots x_{j_{i-1}}}$.

Consider the equation $g(\frac{Z_{j_1, \dots, j_{i-1}}}{x_{j_1}^{e_1} \dots x_{j_{i-1}}^{e_{i-1}}}) - \frac{r_{j_1, \dots, j_{i-1}}}{x_{j_1}^{e_1} \dots x_{j_{i-1}}^{e_{i-1}}} = 0$ where $Z_{j_1, \dots, j_{i-1}}$ is a variable. Multiplying this equation by $(x_{j_1}^{e_1} \dots x_{j_{i-1}}^{e_{i-1}})^{p^s}$ produces a monic polynomial equation in $Z_{j_1, \dots, j_{i-1}}$ with coefficients in R . Let $z_{j_1, \dots, j_{i-1}} \in \overline{K}$ be a root of this equation and let R'' be the R -subalgebra of \overline{K} generated by all the $z_{j_1, \dots, j_{i-1}}$ s, i.e. by the set $\{z_{j_1, \dots, j_{i-1}} \mid 1 \leq j_1 < \dots < j_{i-1} \leq d\}$. Since each $z_{j_1, \dots, j_{i-1}}$ is integral over R and there are finitely many $z_{j_1, \dots, j_{i-1}}$ s, the R -algebra R'' is finite as an R -module.

Let $\tilde{\alpha} = (\frac{z_{j_1, \dots, j_{i-1}}}{x_{j_1}^{e_1} \dots x_{j_{i-1}}^{e_{i-1}}}) \in C^{i-1}(R'')$. The natural inclusion $R \rightarrow R''$ makes $C^\bullet(R)$ into a subcomplex of $C^\bullet(R'')$ in a natural way, and we identify $\tilde{\alpha} \in C^i(R)$ and $\beta \in C^{i-1}(R)$ with their natural images in $C^i(R'')$ and $C^{i-1}(R'')$ respectively. With this identification, $\tilde{\alpha} \in C^i(R'')$ is a cycle representing the image of α under the natural map $H_I^i(R) \rightarrow H_I^i(R'')$, and so is $\bar{\alpha} = \tilde{\alpha} - d_{i-1}(\tilde{\alpha}) \in C^i(R'')$. Since $g(\tilde{\alpha}) = \beta$ and $g(\tilde{\alpha}) = d_{i-1}(\beta)$, we conclude that $g(\bar{\alpha}) = 0$. Let $\bar{\alpha} = (\rho_{j_1, \dots, j_i})$ where $\rho_{j_1, \dots, j_i} \in R''_{x_{j_1} \dots x_{j_i}}$. Each individual ρ_{j_1, \dots, j_i} satisfies the equation $g(\rho_{j_1, \dots, j_i}) = 0$. Since $g(T)$ is a monic polynomial in T with coefficients in R , each ρ_{j_1, \dots, j_i} is an element of the fraction field of R'' that is integral over R . Let R' be obtained from R'' by adjoining all the ρ_{j_1, \dots, j_i} .

Each $\rho_{j_1, \dots, j_i} \in R'$, so the image of α in $H_I^i(R')$ is represented by the cycle $\bar{\alpha} = (\rho_{j_1, \dots, j_i}) \in C^i(R')$ which has all its components ρ_{j_1, \dots, j_i} in R' . Each $R'_{x_{j_1} \dots x_{j_i}}$ contains a natural copy of R' , namely, the one generated by the element $1 \in R'_{x_{j_1} \dots x_{j_i}}$. There is a subcomplex of $C^\bullet(R')$ that in each degree is the direct sum of all such copies of R' . This subcomplex is exact because its cohomology groups are the cohomology groups of R' with respect to the unit ideal. Since $\bar{\alpha}$ is a cycle and belongs to this exact subcomplex, it is a boundary, hence it represents the zero element in $H_I^i(R')$. \square

Corollary 2.3. *Let R be a commutative Noetherian local domain containing a field of characteristic $p > 0$. Assume that R is a surjective image of a Gorenstein local ring. Then the following hold:*

- (a) $H_{\mathfrak{m}}^i(R^+) = 0$ for all $i < \dim R$, where \mathfrak{m} is the maximal ideal of R .

(b) Every system of parameters of R is a regular sequence on R^+ .

Proof. (a) R^+ is the direct limit of the finitely generated R -subalgebras R' , hence $H_{\mathfrak{m}}^i(R^+) = \varinjlim H_{\mathfrak{m}}^i(R')$. But Theorem 2.1 implies that for each R' there is R'' such that the map $H_{\mathfrak{m}}^i(R') \rightarrow H_{\mathfrak{m}}^i(R'')$ in the inductive system is zero. Hence the limit is zero.

(b) Let x_1, \dots, x_d be a system of parameters of R . We prove that x_1, \dots, x_j is a regular sequence on R^+ by induction on j . The case $j = 1$ is clear, since R^+ is a domain. Assume that $j > 1$ and x_1, \dots, x_{j-1} is a regular sequence on R^+ . Set $I_t = (x_1, \dots, x_t)$. The fact that $H_{\mathfrak{m}}^i(R^+) = 0$ for all $i < d$ and the short exact sequences

$$0 \rightarrow R^+/I_{t-1}R^+ \xrightarrow{\text{mult by } x_t} R^+/I_{t-1}R^+ \rightarrow R^+/I_tR^+ \rightarrow 0$$

for $t \leq j-1$ imply by induction on t that $H_{\mathfrak{m}}^q(R^+/(x_1, \dots, x_t)R^+) = 0$ for $q < d-t$. In particular, $H_{\mathfrak{m}}^0(R^+/(x_1, \dots, x_{j-1})R^+) = 0$ since $0 < d-(j-1)$. Hence \mathfrak{m} is not an associated prime of $R^+/(x_1, \dots, x_{j-1})R^+$. This implies that the only associated primes of $R^+/(x_1, \dots, x_{j-1})R^+$ are the minimal primes of $R/(x_1, \dots, x_{j-1})R$. Indeed, if there is an embedded associated prime, say P , then P is the maximal ideal of the ring R_P whose dimension is bigger than $j-1$ and P is an associated prime of $(R^+/(x_1, \dots, x_{j-1})R^+)_P = (R_P)^+/(x_1, \dots, x_{j-1})(R_P)^+$ which is impossible by the above. Hence every element of \mathfrak{m} not in any minimal prime of $R/(x_1, \dots, x_{j-1})R$, for example, x_j , is a regular element on $R^+/(x_1, \dots, x_{j-1})R^+$. \square

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